Characters of the Unitarizable Highest Weight Modules over the N=2 Superconformal Algebras [1]

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The *N=2 superconformal* (or *super-Virasoro*) *algebras* in two dimensions are three complex Lie superalgebras: the *Neveu-Schwarz* superalgebra [1], the *Ramond* superalgebra [1], the *twisted* superalgebra [2], which are denoted as $\mathcal{A}$, $\mathcal{P}$, $\mathcal{T}$, resp., or $\mathcal{G}$ when a statement holds for all three superalgebras. They have the following nontrivial super-Lie brackets :

$$[L_m, L_n] = (m-n)\, L_{m+n} + \tfrac{1}{4} z \left(m^3 - m\right) \delta_{m,-n} \tag{1a}$$

$$[L_m, G^j_n] = (\tfrac{1}{2} m - n)\, G^j_{m+n}\ , \quad j = 1,2 \tag{1b}$$

$$[L_m, Y_n] = -\, n\, Y_{m+n}\ , \qquad [Y_m, Y_n] = z\, m\, \delta_{m,-n} \tag{1c}$$

$$[Y_m, G^j_n] = i\, \epsilon^{jk}\, G^k_{m+n}\ , \quad \epsilon^{jk} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1d}$$

$$[G^j_m, G^k_n]_+ = 2\, \delta^{jk}\, L_{m+n} + i\, \epsilon^{jk}\, (m-n)\, Y_{m+n} + z \left(m^2 - \tfrac{1}{4}\right) \delta^{jk}\, \delta_{m,-n} \tag{1e}$$

where $m \in \mathbb{Z}$ in L_m for all superalgebras; $m \in \mathbb{Z}$ in Y_m and $m \in \frac{1}{2} + \mathbb{Z}$ in G^j_m for $\mathcal{A}$; $m \in \mathbb{Z}$ in Y_m and G^j_m for $\mathcal{P}$; $m \in \mathbb{Z}$ in G^1_m and $m \in \frac{1}{2} + \mathbb{Z}$ in Y_m and G^2_m for $\mathcal{T}$.

The standard triangular decomposition of $\mathcal{G}$ is:

$$\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{H} \oplus \mathcal{G}_- \tag{2}$$

$$\mathcal{H} = \text{l.s.}\{z\,, L_0\,,\ Y_0\,\} \quad \text{for } \mathcal{A}, \mathcal{P} \tag{3a}$$

$$= \text{l.s.}\{z\,, L_0\,,\ G^1_0\,\} \quad \text{for } \mathcal{T}, \quad (G^1_0)^2 = L_0 - z/8 \tag{3b}$$

$$\mathcal{G}_+ = \text{l.s.}\{L_m\,, m > 0,\ Y_n\,, n > 0,\ G^j_p\ \ p > 0\} \ \oplus\ \text{l.s.}\{\bar{G}_0\}_{\mathcal{P}} \tag{4a}$$

$$\mathcal{G}_- = \text{l.s.}\{L_m\,, m < 0,\ Y_n\,, n < 0,\ G^j_p\ \ p < 0\} \ \oplus\ \text{l.s.}\{G_0\}_{\mathcal{P}} \tag{4b}$$

where the generators $G_0\,, \bar{G}_0$ which appear for $\mathcal{P}$ in (4) are the zero modes of:

$$G_n = \tfrac{1}{2}\big(G^1_n + iG^2_n\big)\ , \qquad \bar{G}_n = \tfrac{1}{2}\big(G^1_n - iG^2_n\big) \tag{5}$$

[1] This is a slightly extended version of an Encyclopedia entry.

A highest weight module (HWM) over $\mathcal{G}$ is characterized by its highest weight $\lambda \in \mathcal{H}^*$ and highest weight vector v_0 so that $X\, v_0 = 0$, for $X \in \mathcal{G}_+$, $H\, v_0 = \lambda(H) v_0$ for $H \in \mathcal{H}$. Denote $\lambda(L_0) = h$, $\lambda(z) = c$, $\lambda(Y_0) = q$. [Note that interchanging G_0 and $\bar{G}_0$ in (4) means to pass from P^+ to P^- modules in the terminology of [2].] The largest HWM with these properties is the Verma module $V^\lambda = V^{h,c,q}$ ($= V^{h,c}$ for $\mathcal{T}$), which is isomorphic to $U(\mathcal{G}_-)v_0$, where $U(\mathcal{G}_-)$ denotes the universal enveloping algebra of $\mathcal{G}_-$. Denote by L^λ (resp. $L^{h,c,q}$, $L^{h,c}$) the factor-module V^λ / I^λ, where I^λ is the maximal proper submodule of V^λ. Then every irreducible HWM over $\mathcal{G}$ is isomorphic to some L^λ.

A Verma module $V^{h,c,q}$ ($V^{h,c}$) over $\mathcal{G}$ is reducible if and only if [2]:

$$f^A_{r,s} \equiv 2h(c-1) - q^2 - \tfrac{1}{4}(c-1)^2 + \tfrac{1}{4}[(c-1)r + s]^2 = 0, \quad \text{for some } r \in I\!N, s \in 2I\!N, \tag{6a}$$

$$\textit{or} \quad g^A_n \equiv 2h - 2nq + (c-1)(n^2 - \tfrac{1}{4}) = 0, \quad \text{for some } n \in \tfrac{1}{2} + Z\!\!Z, \quad \text{for } \mathcal{A}\ ; \tag{6b}$$

$$f^P_{r,s} \equiv 2(c-1)(h - \tfrac{1}{8}) - q^2 + \tfrac{1}{4}[(c-1)r + s]^2 = 0, \quad \text{for some } r \in I\!N, s \in 2I\!N, \tag{7a}$$

$$\textit{or} \quad g^P_n \equiv 2h - 2nq + (c-1)(n^2 - \tfrac{1}{4}) - \tfrac{1}{4} = 0, \quad \text{for some } n \in Z\!\!Z, \quad \text{for } \mathcal{P}\ ; \tag{7b}$$

$$f^T_{r,s} \equiv 2(c-1)(h - \tfrac{1}{8}) + \tfrac{1}{4}[(c-1)r + s]^2 = 0, \quad \text{for some } r \in I\!N, s \in 2I\!N - 1, \quad \text{for } \mathcal{T} \tag{8}$$

The necessary conditions for the unitarity of $L^{h,c,q}$ ($L^{h,c}$) are [2]:

$$\text{case } A_3 : \quad c \geq 1,\ g^A_n \geq 0, \quad \text{for all } n \in \tfrac{1}{2} + Z\!\!Z\ ; \tag{9a}$$

$$\text{case } A_2 : \quad c \geq 1,\ f^A_{1,2} \geq 0,\ g^A_n = 0,\ g^A_{n+\mathrm{sign}(n)} \leq 0, \quad \text{for some } n \in \tfrac{1}{2} + Z\!\!Z\ ; \tag{9b}$$

$$\text{case } A_0 : \quad c < 1,\ c = 1 - \tfrac{2}{m},\quad h = \tfrac{1}{m}(jk - \tfrac{1}{4}),\quad q = \tfrac{1}{m}(j-k),$$
$$\text{for } \ m \in 1 + I\!N, \quad j,k \in \tfrac{1}{2} + Z\!\!Z, \quad 0 < j,k,j+k \leq m-1\ ; \tag{9c}$$

$$\text{case } P_3 : \quad c \geq 1,\ g^P_n \geq 0, \quad \text{for all } n \in Z\!\!Z\ ; \tag{10a}$$

$$\text{case } P_2 : \quad c \geq 1,\ f^P_{1,2} \geq 0,\ g^P_n = 0,\ g^P_{n+\mathrm{sign}(n)} < 0, \quad \text{for some } n \in Z\!\!Z\ , \tag{10b}$$
$$\mathrm{sign}(0) = \pm 1 \text{ for } P^\pm\ ;$$

$$\text{case } P_0 : \quad c < 1,\ c = 1 - \tfrac{2}{m},\quad h = \tfrac{1}{8}c + \tfrac{jk}{m},\quad q = \pm\tfrac{1}{m}(j-k),$$
$$\text{for } \ m \in 1 + I\!N, \quad j,k \in Z\!\!Z, \quad 0 \leq j-1,k,j+k \leq m-1\ ; \tag{10c}$$

$$\text{case } T_2 : \quad c \geq 1, \quad h \geq \tfrac{1}{8}c\ ; \tag{11a}$$

$$\text{case } T_0 : \quad c < 1,\ c = 1 - \tfrac{2}{m},\quad h = \tfrac{1}{8}c + \tfrac{1}{16m}(m-2r)^2,$$
$$\text{for } \ m \in 1 + I\!N, \quad r \in I\!N, \quad 1 \leq r \leq \tfrac{1}{2}m\ ; \tag{11b}$$

Further write $V^{h,c,(q)}, L^{h,c,(q)}$ in the cases when a statement holds for $V^{h,c,q}, L^{h,c,q}$ over $\mathcal{A}, \mathcal{P}$ as written and for $V^{h,c}, L^{h,c}$ over $\mathcal{T}$ after deleting q and all related quantities.

The weight decomposition of $V^{h,c,(q)}$ is:

$$V^{h,c,(q)} = \bigoplus_{n,(m)} V^{h,c,(q)}_{n,(m)} \tag{12a}$$

$$V^{h,c,(q)}_{n,(m)} = \{ \ v \in V^{h,c,(q)} \mid L_0\, v = (h+n) v, \ \text{ for } \mathcal{G} \ , \quad Y_0\, v = (q+m) v, \ \text{ for } \mathcal{A}\,, \mathcal{P} \ \} \tag{12b}$$

where the ranges of n, m in (12) are:

$$n \in \tfrac{1}{2} \mathbb{Z}_+ \ , \quad m \in 2n + 2\mathbb{Z}\,, \quad |m| \le \sqrt{2n}\,, \quad \text{for } \mathcal{A} \tag{13a}$$

$$n \in \mathbb{Z}_+ \ , \quad m \in \mathbb{Z}\,, \quad |\tfrac{1}{2}(1-\sqrt{8n+1}) \le m \le \tfrac{1}{2}(1-\sqrt{8n+1})\,, \quad \text{for } \mathcal{P} \tag{13b}$$

$$n \in \tfrac{1}{2} \mathbb{Z}_+ \ , \quad \text{for } \mathcal{T} \tag{13c}$$

n is called the level of $V^{h,c,(q)}_{n,(m)}$, m - its relative charge.

Then the character of $V^{h,c,(q)}$ may be defined as follows [2]:

$$\mathrm{ch}\, V^{h,c,q} = \sum_{n,m} (\dim \ V^{h,c,q}_{n,m})\, x^{h+n}\, y^{q+m} = \sum_{n,m} P(n,m)\, x^{h+n}\, y^{q+m} = x^h\, y^q\, \psi(x,y) \tag{14a}$$

$$\mathrm{ch}\, V^{h,c} = \sum_{n} (\dim \ V^{h,c}_{n})\, x^{h+n} = \sum_{n} P_T(n)\, x^{h+n} = x^h\, \psi_T(x) \ , \tag{14b}$$

$$\psi_A(x,y) \equiv \sum_{n,m} P_A(n,m)\, x^n\, y^m = \prod_{k\in\mathbb{N}} \frac{(1+x^{k-1/2}y)(1+x^{k-1/2}y^{-1})}{(1-x^k)^2} \tag{15a}$$

$$\psi_P(x,y) \equiv \sum_{n,m} P_P(n,m)\, x^n\, y^{m-1/2} = (y^{1/2}+y^{-1/2}) \prod_{k\in\mathbb{N}} \frac{(1+x^k y)(1+x^k y^{-1})}{(1-x^k)^2} \tag{15b}$$

$$\psi_T(x) \equiv \sum_{n} P_T(n)\, x^n = \prod_{k\in\mathbb{N}} \frac{(1+x^k)(1+x^{k-1/2})}{(1-x^k)(1-x^{k-1/2})} \tag{15c}$$

(for P^- representations one should write $y^{m+1/2}$ instead of $y^{m-1/2}$ [2]).

Proposition 1: [2],[3] The character formulae for the unitary cases A_3, (P_3), with either $c > 1$ and $g_n > 0$, $\forall n \in \frac{1}{2} + \mathbb{Z}$, ($\forall n \in \mathbb{Z}$), or $c = 1$, and cases T_2 are given by:

$$\mathrm{ch}\, L^{h,c,q} = \mathrm{ch}\, V^{h,c,q} \tag{16a}$$

$$\mathrm{ch}\, L^{h,c} = \mathrm{ch}\, V^{h,c} \ , \quad h \neq \tfrac{c}{8} \ , \qquad \mathrm{ch}\, L^{\frac{c}{8},c} = \tfrac{1}{2}\, \mathrm{ch}\, V^{\frac{c}{8},c} \tag{16b}$$

Note that the Verma modules involved are irreducible except in the last case, where $V^{\frac{c}{8},c} = I^{\frac{c}{8},c} \oplus V^{\frac{c}{8},c}/I^{\frac{c}{8},c}$, $\quad I^{\frac{c}{8},c} \cong V^{\frac{c}{8},c}/I^{\frac{c}{8},c}$. $\diamondsuit$

Proposition 2: [3] The character formulae for the unitary cases A_3, (P_3), with $c > 1$, $q/(c-1) = n_0 \in \frac{1}{2} + \mathbb{Z}$, $(n_0 \in \mathbb{Z})$, and $g_{n_0} = 0$, and for the cases A_2, (P_2), with $f_{1,2} > 0$, are given by:

$$\mathrm{ch}\, L^{h,c,q} \;=\; \widetilde{\mathrm{ch}}_n \; V^{h,c,q} \;\equiv\; \frac{1}{(1 + x^{|n|} y^{\mathrm{sign}(n)})}\; \mathrm{ch}\, V^{h,c,q} \tag{17}$$

where for A_3, P_3, $n = n_0$, and for A_2, P_2, n is such that $g_n = 0$, $g^A_{n+\mathrm{sign}(n)} < 0$.

Proof: Actually, the Proposition holds in a more general situation beyond the unitary cases, namely, when for a fixed $V^{h,c,q}$ $(6b)$, $((7b))$ holds for some n, possibly also for some n' such that $\mathrm{sign}(n) = \mathrm{sign}(n')$ and $|n'| > |n|$, and $(6a)$, $((7a))$ does not hold for any r, s. [In the statement of Proposition 2 the additional reducibility appears in the cases A_2, (P_2) when $2q(c-1) \in \mathbb{Z}$, then $n' = M - n$, $M \equiv 2q(c-1)$ and $g_{M-n} = 0$.] In this situation there is a singular vector v^s_n and possibly a singular vector $v^s_{n'}$, however, the latter (when existing) is a descendant of v^s_n. Thus, there is the following embedding diagram:

$$V^{h,c,q} \;\longrightarrow\; V^{h+|n|,c,q+\mathrm{sign}(n)} \tag{18}$$

where is used the convention that the arrow points *to* the embedded module. This embedding has a kernel, since there is an infinite chain of embeddings of Verma modules:

$$\cdots \;\longrightarrow\; V_t \;\longrightarrow\; V_{t+1} \;\longrightarrow\; \cdots \tag{19}$$

where $V_t \equiv V^{h+t|n|,c,q+t\mathrm{sign}(n)}$, $t \in \mathbb{Z}$. Using the Grassmannian properties of the odd generators one can show that this chain of embedding maps is exact. Due to the kernel one has:

$$\mathrm{ch}\, L^{h,c,q} \;=\; \mathrm{ch}\, V^{h,c,q} \;-\; \widetilde{\mathrm{ch}}_n\, V^{h+|n|,c,q+\mathrm{sign}(n)} \;=\; \widetilde{\mathrm{ch}}_n\, V^{h,c,q} \tag{20}$$

Proposition 3: [3] The character formulae for the unitary cases A_2, P_2, with $f_{1,2} = 0$ is given by:

$$\mathrm{ch}\, L^{h,c,q} \;=\; \frac{(1-x)}{(1 + x^{|n|} y^{\mathrm{sign}(n)})\,(1 + x^{|n|+1} y^{\mathrm{sign}(n)})}\; \mathrm{ch}\, V^{h,c,q} \tag{21}$$

Proof: The character relevant structure of $V^{h,c,q}$ is given by the embedding diagram:

$$\begin{tikzcd}
 & V_0 = V^{h,c,q} \arrow[dl, "/" marking] \arrow[dr] & \\
V'_0 = V^{h+1,c,q} \arrow[dr] & & V_1 = V^{h+|n|,c,q+\mathrm{sign}(n)} \arrow[dl, "/" marking] \\
 & V'_1 = V^{h+|n|+2,c,q+\mathrm{sign}(n)} &
\end{tikzcd} \tag{22}$$

where the dashed arrows denote even embeddings, V_0 is reducible v.r.t. $g_n = 0 = f_{1,2}$ from the statement; then (with $\mu = 0$ for $\mathcal{A}$, $\mu = 1$ for $\mathcal{P}$) :

$$h = \tfrac{1}{8}(c-1)(2n+\epsilon)^2 + n\epsilon + \tfrac{1}{8}\mu \ , \qquad q = \tfrac{1}{2}(c-1)(2n+\epsilon) \ , \qquad \epsilon \equiv \mathrm{sign}(n) \tag{23}$$

The other reducibilities relevant for the structure are: V_1 w.r.t. $g_n = 0 = f_{1,2}$, V_0' and V_1' w.r.t. $g_{n+\mathrm{sign}(n)} = 0$. Thus for the character formula follows:

$$\mathrm{ch}\, L^{h,c,q} \ = \ \mathrm{ch}\, V^{h,c,q} - \mathrm{ch}\, V^{h+1,c,q} - \widetilde{\mathrm{ch}}_n\, V^{h+|n|,c,q+\mathrm{sign}(n)} + \widetilde{\mathrm{ch}}_{n+\mathrm{sign}(n)}\, V^{h+|n|+2,c,q+\mathrm{sign}(n)} \tag{24}$$

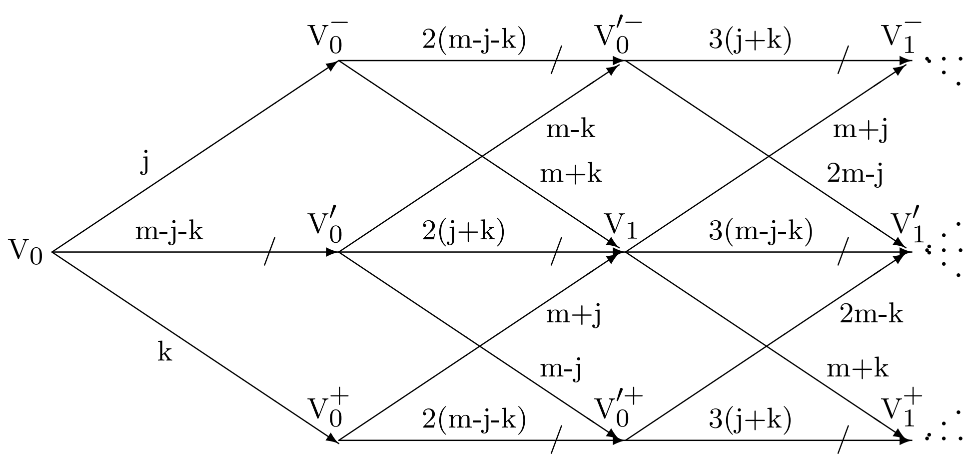
which after substituting the definitions gives (21). $\diamondsuit$

Proposition 4: [3],[4],[5] The character formulae for the unitary cases A_0, $P_0^{\pm}$, is given by:

$$\begin{aligned}\mathrm{ch}\, L_{m,j,k}(x,y) \ = \ \sum_{n\in\mathbb{Z}_+} & x^{mn^2+(j+k)n} \Big\{ \ 1 - x^{(m-j-k)(2n+1)} \ + \\ & + \ x^{mn+k} y \Big[\frac{x^{2(m-j-k)(n+1)}}{1+x^{mn+m-j}y} - \frac{1}{1+x^{mn+k}y} \Big] + \\ & + \ x^{mn+k} y^{-1} \Big[\frac{x^{2(m-j-k)(n+1)}}{1+x^{mn+m-k}y^{-1}} - \frac{1}{1+x^{mn+j}y^{-1}} \Big] \Big\} \ \mathrm{ch}\, V_{m,j,k}(x,y)\end{aligned} \tag{25}$$

where $L_{m,j,k} = L^{h,c,q}$, $V_{m,j,k} = V^{h,c,q}$, when h, c, q are expressed through m, j, k as in (9c), (10c).

Proof: The structure of $V_0 \equiv V_{m,j,k}$ is given by the following embedding diagram:



(26)

$$\begin{aligned} & V_n = V^{h+mn^2+(k+j)n,c,q} \ , \quad V_n' = V^{h+mn^2+m(2n+1)-(k+j)(n+1),c,q} \ , \\ & V_n^+ = V^{h+mn^2+(m+k+j)n+k,c,q+1} \ , \quad V_n'^+ = V^{h+mn^2+m(3n+2)-(k+j)(n+2)+k,c,q+1} \ , \\ & V_n^- = V^{h+mn^2+(m+k+j)n+j,c,q-1} \ , \quad V_n'^- = V^{h+mn^2+m(3n+2)-(k+j)(n+2)+j,c,q-1} \end{aligned} \tag{27}$$

From this follows:

$$\mathrm{ch}\,L_{m,j,k}(x,y) \;=\; \sum_{n\in\mathbb{Z}_+}\Big[\mathrm{ch}\,V_n \;-\; \mathrm{ch}\,V'_n \;-\; \widetilde{\mathrm{ch}}_{mn+k}\,V_n^+ \;-\; \widetilde{\mathrm{ch}}_{mn+j}\,V_n^- \;+ \\ +\; \widetilde{\mathrm{ch}}_{mn+m-j}\,V_n'^+ \;+\; \widetilde{\mathrm{ch}}_{mn+m-k}\,V_n'^-\Big] \tag{28}$$

which after substituting the definitions gives (25). $\diamondsuit$

Remark: It should be stressed that diagram (26) is used only as representing the structure of the Verma module $V_{m,j,k}$. In particular, later it was shown that each even embedding between the Verma modules V_n and V'_n, $n = 1, 2, \ldots$, and between the Verma modules V'_n and V_{n+1}, $n = 1, 2, \ldots$, is generated by two uncharged fermionic singular vectors [6]. However, this has no relevance for the character formulae.

Proposition 5: [3],[4],[5] Let $V_{r,s}$, $r \in \mathbb{N}$, $s \in \mathbb{N} - 1/2$, be the Verma module $V^{h,c}$ with $h = h^T_{r,s} = [(tr - ms)^2 - t^2]/4mt + 1/8 = h^T_{m-r,t-s}$, $c = 1 - 2t/m$, $t, m \in \mathbb{N}$, $tr \le ms$, $s < t < m$, t, m have no common divisor. Then the character formula for the corresponding irreducible quotient $L_{r,s}$ is given by:

$$\mathrm{ch}\,L_{r,s}(x) \;=\; \mathrm{ch}\,V_{r,s}(x)\sum_{j\in\mathbb{Z}} x^{j(tmj+tr-ms)}(1 - x^{s(2mj+r)}) \tag{29}$$

In particular, the character formula for the T_0 unitary cases $r \le m/2$ is obtained from (29) by setting $t = 1$, $s = 1/2$.

The Proof relies on the realization that the Verma modules $V_{r,s}$ has exactly the structure of certain Virasoro and $N = 1$ super-Virasoro (Neveu-Schwarz and Ramond) Verma modules for which the character formulae were known (see also the corresponding encyclopedia entry). $\diamondsuit$

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